PHYSICAL REVIEW E **72**, 056114 (2005)

Universality of fluctuation-dissipation ratios: The ferromagnetic model

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(Received 11 August 2005; published 11 November 2005)

We calculate analytically the fluctuation-dissipation ratio (FDR) for Ising ferromagnets quenched to criticality, both for the long-range model and its short-range analog in the limit of large dimension. Our exact solution shows that, for both models, $X^{\infty}=1/2$ if the system is unmagnetized while $X^{\infty}=4/5$ if the initial magnetization is nonzero. This indicates that two different classes of critical coarsening dynamics need to be distinguished depending on the initial conditions, each with its own nontrivial FDR. We also analyze the dependence of the FDR on whether local and global observables are used. These results clarify how a proper local FDR (and the corresponding effective temperature) should be defined in long-range models in order to avoid spurious inconsistencies and maintain the expected correspondence between local and global results; global observables turn out to be far more robust tools for detecting nonequilibrium FDRs.

DOI: 10.1103/PhysRevE.72.056114 PACS number(s): 05.70.Jk, 02.50.−r, 05.20.−y, 05.40.−a

I. INTRODUCTION

One of the main goals of modern statistical mechanics is to find a general theory of nonequilibrium processes. Although significant advances have been made in the past [1,2], a complete theory of nonequilibrium systems analogous to thermodynamics does not yet exist. A common approach has been to extend well-known equilibrium concepts to the nonequilibrium regime. One of the most important among these is the temperature, and many researchers have therefore tried to extend it to systems out of equilibrium by introducing a so-called effective temperature [3]. The question of whether and how such a quantity can be defined properly is a central issue in the construction of a general theory of nonequilibrium systems. Different definitions have been employed in granular systems [4], driven systems [5], and glassy systems [6] among others. Studies of mean-field spin glasses have shown that an effective temperature can be defined by measuring the violation of the fluctuation-dissipation theorem (FDT) in terms of a so-called fluctuation-dissipation ratio (FDR) [7]

$$X(t,t_{\mathrm{w}}) = \frac{TR(t,t_{\mathrm{w}})}{\partial C(t,t_{\mathrm{w}})/\partial t_{\mathrm{w}}}$$

where T is the temperature of the heat bath, $C(t,t_{\mathrm{w}})$ is the autocorrelation function of a given observable, and $R(t,t_{\mathrm{w}})$ is the conjugate response function; the latter encodes the change of the value of the observable at time t to a small perturbation at an earlier time t_{w}. In equilibrium the FDT is verified and $X(t,t_{\mathrm{w}})=1$. Whether the FDR is useful more generally, and in particular beyond mean-field models, has been the subject of debate in recent years [8,9]. It has been shown that the effective temperature defined through the FDR, $T_{\mathrm{eff}}(t,t_{\mathrm{w}})=T/X(t,t_{\mathrm{w}})$, has good thermodynamic properties for some mean-field models [6]; the zeroth law of thermodynamics can also be extended to the nonequilibrium regime [10]. Nevertheless, there are still many open questions regarding the physical meaning and universality of the FDR [11,12].

An important aspect in the study of FDT violation in glassy systems is its application to ferromagnetic systems which are quenched from high temperature to the critical temperature (see, e.g., Refs. [13–15] and the recent review [16]) or below. The ensuing nonequilibrium evolution, where the system coarsens—by the growth of domains with the equilibrium magnetization, for $T<T_c$—is of course different from that of glasses in many respects; for example, thermal activation effects are irrelevant for the long-time dynamics. However, there are also appealing similarities. In particular, equilibrium is never reached in an infinite system and as a consequence the system exhibits aging, i.e., a dependence of the relaxation properties on the time elapsed since the quench.

The simplest ferromagnetic model that can be studied at criticality is the Ising chain first solved by Glauber [17]. At the critical point, $T=0$, the magnetization jumps discontinuously from 0 ($T>0$) to 1. The relaxation dynamics at zero temperature after a quench from $T=\infty$, i.e., a random initial configuration, has been studied in, e.g., Refs. [18–20]. Recently, the FDR has been calculated analytically for a randomly staggered perturbation and the corresponding spin autocorrelation function [21,22]. For long times t and t_{w} this gives $X(t,t_{\mathrm{w}})=(1+t_{\mathrm{w}}/t)/2$. In the limit $t\to\infty$ the FDR then approaches $X=1/2$, which coincides with the value obtained in models characterized by diffusive dynamics (such as the random walk or the Gaussian model [23]). These results have led to the suggestion [13,24] that for systems at criticality the limiting value of the FDR

$$X^{\infty} = \lim_{t_{\mathrm{w}}\to\infty}\lim_{t\to\infty} X(t,t_{\mathrm{w}})$$

is a universal quantity. Consistent with this, the exact solution of the ferromagnetic spherical model in d dimensions at criticality also gives $X^{\infty}=1/2$ for $d>4$, i.e., above the upper critical dimension [13,24].

In considering the universality of the limiting FDR, one issue is whether and how the limiting FDR depends on the observable whose correlation and response are measured. An obvious alternative to the local spin autocorrelation is its long-wavelength analog, i.e., the correlation function of the fluctuating magnetization. Exact calculations for the Ising chain [14,25] and the spherical model [26] as well as numerical simulations [14,27] for the Ising model in $d=2$ show that the resulting global X^{∞} is always identical to the local version. This local-global correspondence, which can also be obtained by field-theoretic arguments [15,16], is rather reassuring; physically, it arises because the long wavelength Fourier components of the spins are slowest to relax and dominate the long-time behavior of both local and global quantities. For a numerical determination of the limiting FDR the global quantities are often more suitable [14,27] because plots of susceptibility (integrated response) versus correlation are close to straight lines with slope X^{∞}. We do not discuss in this paper the FDR for other observables that are nonlinear in the spins, e.g., the energy; this question is studied in [15,16,26].

A further key question is to what extent the limiting FDR X^{∞} depends on the initial conditions for the coarsening dynamics. The results quoted above all apply to initial high-temperature equilibrium, i.e., an unmagnetized system with no or only short-range spatial correlations. Some initial progress in considering more general initial states has already been made. For the Ising chain [28] a nonzero initial magnetization does not change the value of X^{∞}; this is unlikely to be a general result, however, because the Ising chain at its $T=0$ critical point has the peculiarity that the magnetization remains constant instead of decaying to zero. Other values of the limiting FDR are nevertheless possible even in the Ising chain; e.g., when the magnetization is zero initially but correlations between spins are so strong that only a finite number of domain walls exist in the system, one finds [29] $X^{\infty}=0$. A more general analysis for unmagnetized but spatially correlated initial conditions in the spherical model gives qualitatively similar results [28]: if initial correlations are strong (i.e., decay with distance as slow power laws), the limiting FDR is $X^{\infty}=0$; otherwise it is the same as for an uncorrelated initial state.

In this paper we investigate in full detail the dependence of the FDR on the initial condition for a ferromagnet with long-range interactions as well as short-range interactions in the limit of large dimension d. Our main result will be that the nonequilibrium dynamics starting from unmagnetized and magnetized initial conditions are in different universality classes that are distinguished by different and nontrivial, i.e., nonzero, values of X^{∞}. We also analyze the correspondence between local and global FDRs. In a naive analysis this appears broken in the long-range case, but we show that it is recovered when finite-size corrections are included.

The paper is organized as follows. Section II describes in outline the calculation of the basic evolution equations for correlation and response in the long-range ferromagnet. In Sec. III we compute from these the global and local FDRs. Section IV contains the corresponding analysis for the short-range ferromagnet, where the length-scale dependence of the FDR can be made explicit and the local-global correspondence holds as expected. In Sec. V this correspondence is shown to hold also for the long-range model once finite-size corrections are accounted for. Section VI, finally, summarizes our key results and conclusions. Technical details are relegated to two Appendices.

II. LONG-RANGE FERROMAGNET

In this section we study the ferromagnet with long-range interactions and in particular the nonequilibrium dynamics of the relaxation functions (correlations and responses). We leave technical details to Appendix A and focus here on the conceptual aspects of the calculation and the results.

The model is defined by the Hamiltonian

$$H=-\frac{J_0}{N-1}\sum_{i,j}\sigma_i\sigma_j-\sum_i h_i^{\mathrm{ext}}\sigma_i \tag{1}$$

where the $\sigma_i=\pm 1$, $i=1,\ldots,N$, denote Ising spin variables and h_i^{ext} is a position-dependent external field. The strength of the coupling between spins, J_0, is normalized by a factor $1/(N-1)$ to ensure an extensive energy; we will take $J_0=1$ without loss of generality. The dynamics we consider is of Glauber type: each spin σ_i flips independently with rate $[1-\sigma_i\tanh(\beta h_i)]/2$ where $\beta=1/T$ is the inverse temperature and h_i the local field acting on spin i,

$$h_i=h_i^{\mathrm{ext}}+\frac{1}{N-1}\sum_{j\neq i}\sigma_j. \tag{2}$$

To keep the notation compact we define the function $\mathrm{th}(z)=\tanh(\beta z)$ and abbreviate $t_i=\tanh(\beta h_i)=\mathrm{th}(h_i)$. Multiplying the flip rate by the change $-2\sigma_i$ when σ_i flips, it follows that

$$\frac{\partial}{\partial t}\langle\sigma_i\rangle=\langle t_i-\sigma_i\rangle \tag{3}$$

where the angular brackets denote an average over the thermal history of the system and over the initial conditions. For spin products $(i\neq j)$ one finds similarly

$$\frac{\partial}{\partial t}\langle\sigma_i\sigma_j\rangle=\langle(t_i-\sigma_i)\sigma_j\rangle+\langle\sigma_i(t_j-\sigma_j)\rangle. \tag{4}$$

Throughout this paper, omitted time arguments indicate dynamical averages evaluated at time t. To obtain the two-time correlation functions we can take advantage of the fact that Eq. (3) does not depend on the initial condition and so is equally valid for correlations with some quantity at an earlier time $t_{\mathrm{w}}<t$. This gives

$$\frac{\partial}{\partial t}\langle\sigma_i(t)\sigma_j(t_{\mathrm{w}})\rangle=\langle[t_i(t)-\sigma_i(t)]\sigma_j(t_{\mathrm{w}})\rangle. \tag{5}$$

Using Eq. (3) we can also express the time evolution of the magnetization, $m=N^{-1}\Sigma_i\langle\sigma_i\rangle$, as

$$\frac{\partial m}{\partial t}=-m+\frac{1}{N}\sum_i\langle t_i\rangle. \tag{6}$$

The relaxation dynamics of physical systems is routinely characterized by correlation and response functions. We will consider here the properties of both global and local relaxation functions. Our main goal is to clarify the relationship between them and the implications for the corresponding FDRs. Note that while the fluctuations of individual spins, as encoded in the local (auto) correlation function, are $O(1)$, those of the fluctuating magnetization $N^{-1}\Sigma_i \sigma_i$ are $O(N^{-1/2})$. Nevertheless we will see that the same physics can be extracted from both quantities. The global correlation function is scaled by a factor of N below to give an $O(1)$ quantity as in the local case.

Previous studies [30,31] have shown that the relevant correlation functions are the connected ones, defined by

$$C_{ij}(t,t_{\rm w}) = \langle \sigma_i(t)\sigma_j(t_{\rm w})\rangle - \langle \sigma_i(t)\rangle\langle \sigma_j(t_{\rm w})\rangle.$$

First we analyze the equal-time correlations. For $i \neq j$ it follows from Eq. (4) that

$$\begin{aligned}\frac{\partial}{\partial t}C_{ij}(t,t) &= \langle (t_i-\sigma_i)\sigma_j\rangle + \langle \sigma_i(t_j-\sigma_j)\rangle \\ &\quad - \langle t_i-\sigma_i\rangle\langle\sigma_j\rangle - \langle\sigma_i\rangle\langle t_j-\sigma_j\rangle \\ &= -2C_{ij}(t,t) + \langle \Delta t_i \Delta\sigma_j\rangle + \langle \Delta\sigma_i \Delta t_j\rangle, \end{aligned} \tag{7}$$

where we use the notation $\Delta\psi = \psi - \langle\psi\rangle$ for the deviation of any quantity from its average, so that, e.g., $\Delta t_i = t_i - \langle t_i\rangle$ and similarly for $\Delta\sigma_i$. For the two-time correlations one gets similarly from Eq. (5)

$$\begin{aligned}\frac{\partial}{\partial t}C_{ij}(t,t_{\rm w}) &= \langle [t_i(t)-\sigma_i(t)]\sigma_j(t_{\rm w})\rangle - \langle [t_i(t)-\sigma_i(t)]\rangle\langle\sigma_j(t_{\rm w})\rangle \\ &= -C_{ij}(t,t_{\rm w}) + \langle \Delta t_i(t)\Delta\sigma_j(t_{\rm w})\rangle. \end{aligned} \tag{8}$$

Equations (7) and (8) are general and also valid for short-range systems provided that the appropriate local field replaces the long-range expression (2).

To make progress, we exploit the fact that in the long-range model, for large N, the correlations between different spins are of $O(N^{-1})$. So the fluctuations Δh_i of h_i around its mean $h_i^{\rm ext}+m$ are small, of $O(N^{-1/2})$. In equilibrium away from criticality one can indeed show that $\langle(\Delta h_i)^2\rangle = O(N^{-1})$ while $\langle(\Delta h_i)^2\Delta\sigma_j\rangle$ and $\langle(\Delta h_i)^3\rangle$ are $O(N^{-2})$. We will assume that the correlations out of equilibrium are of the same order. This is reasonable if we start from an initial state with weak correlations and, for quenches to criticality, also restrict ourselves to the interesting nonequilibrium regime where all times are short compared to the equilibration time.

Setting the external field to zero, we can Taylor expand the nonlinear terms in our equations of motion in powers of Δh_i:

$$t_i = {\rm th}(m) + \Delta h_i {\rm th}'(m) + \frac{1}{2}(\Delta h_i)^2 {\rm th}''(m) + \cdots. \tag{9}$$

Since we are only interested in the leading terms, we truncate this expansion after the linear term in Δh_i; subleading corrections are discussed in Sec. V. Since $\langle\Delta h_i\rangle = 0$, the leading order term in the equation of motion for the magnetization (6) takes the expected mean-field form

$$\frac{\partial m}{\partial t} = -m + {\rm th}(m). \tag{10}$$

Due to spatial translation invariance (or more precisely permutation invariance) between spins in the long-range ferromagnet, there are only two different correlation functions, one local and one nonlocal. We write these as

$$C_{ii} = C_{\rm loc} + O(N^{-1}), \tag{11}$$

$$C_{ij} = C_{\rm nl}/N + O(N^{-2}), \tag{12}$$

using the fact that the nonlocal correlations are only $O(1/N)$ to leading order.

In terms of the local (11) and nonlocal (12) correlations, the global correlation is defined by

$$C_{\rm g} \equiv C_{\rm loc} + C_{\rm nl} \tag{13}$$

and gives the leading contribution of the correlator of the magnetization; see Eq. (A1) of Appendix A. In order to compute the dynamical equation for the global correlation function (13) we need the equations for the local and nonlocal correlation functions, which can be expressed as

$$\frac{\partial}{\partial t}C_{\rm nl}(t,t) = -2aC_{\rm nl}(t,t) + 2{\rm th}'(m)(1-m^2), \tag{14}$$

$$\frac{\partial}{\partial t}C_{\rm nl}(t,t_{\rm w}) = -C_{\rm nl}(t,t_{\rm w}) + {\rm th}'(m)C_{\rm g}(t,t_{\rm w}), \tag{15}$$

$$C_{\rm loc}(t,t) = 1 - m^2(t), \tag{16}$$

$$\frac{\partial}{\partial t}C_{\rm loc}(t,t_{\rm w}) = -C_{\rm loc}(t,t_{\rm w}), \tag{17}$$

as shown in Appendix A [see Eqs. (A2) and (A5)–(A7)]. The quantity a appearing in Eq. (14) is defined as

$$a = 1 - {\rm th}'(m) = 1 - \beta[1-\tanh^2(\beta m)]. \tag{18}$$

We will not normally write its time dependence explicitly.

In order to complete the analysis of the dynamics of our model, we need to find the linear response to applied external fields. This is characterized by the response functions

$$R_{ij}(t,t_{\rm w}) = \frac{\delta\langle\sigma_i(t)\rangle}{\delta h_j^{\rm ext}(t_{\rm w})}.$$

As for the correlation functions, we have to leading order in N

$$R_{ii} = R_{\rm loc} + O(N^{-1}), \tag{19}$$

$$R_{ij} = R_{\rm nl}/N + O(N^{-2}), \tag{20}$$

while the leading term in the global response, of the magnetization to a uniform field, is

$$R_{\rm g} = R_{\rm loc} + R_{\rm nl}. \tag{21}$$

The evolution equations for these response functions can be expressed as

$$\frac{\partial}{\partial t}R_{\mathrm{nl}}(t,t_{\mathrm{w}}) = \mathrm{th}'(m)[R_{\mathrm{loc}}(t,t_{\mathrm{w}}) + R_{\mathrm{nl}}(t,t_{\mathrm{w}})] - R_{\mathrm{nl}}(t,t_{\mathrm{w}}), \tag{22}$$

$$\frac{\partial}{\partial t}R_{\mathrm{loc}}(t,t_{\mathrm{w}}) = - R_{\mathrm{loc}}(t,t_{\mathrm{w}}), \tag{23}$$

as derived in Appendix A [see Eqs. (A10) and (A11)]. These equations can be integrated forward in time starting from the values of the equal-time response functions (A13):

$$R_{\mathrm{loc}}(t,t) = \mathrm{th}'(m), \quad R_{\mathrm{nl}}(t,t) = 0. \tag{24}$$

The evolution equations (14)–(17) for the correlations and (22)–(24) for the responses contain all the relevant information about the dynamical properties of the ferromagnetic systems.

III. FLUCTUATION-DISSIPATION RATIOS

Using the results obtained in the previous section, we can study in full detail the fluctuation-dissipation ratios for global and local relaxation functions. Crucially, we will be able to investigate how the value of the asymptotic global FDR depends on the initial condition. Our calculation in the infinite-system-size limit will produce different values for global and local FDRs; this apparent breaking of the expected local-global correspondence will be solved and carefully explained in Sec. V.

A. Global FDR

To compute the global FDR, we first need the equal-time global correlation $C_{\mathrm{g}}(t,t)=C_{\mathrm{loc}}(t,t)+C_{\mathrm{nl}}(t,t)=1-m^2(t)+C_{\mathrm{nl}}(t,t)$. Differentiating this expression with respect to time and using Eqs. (10) and (14) we get

$$\frac{\partial}{\partial t}C_{\mathrm{g}}(t,t) = -2m[\mathrm{th}(m) - m] - 2aC_{\mathrm{nl}}(t,t) + 2(1-a)(1-m^2)$$
$$= -2aC_{\mathrm{g}}(t,t) + b \tag{25}$$

with $b=2[1-m\mathrm{th}(m)]$ and a defined in Eq. (18). Equation (25) can be integrated explicitly to get

$$C_{\mathrm{g}}(t,t) = r^2(t)C_{\mathrm{g}}(0,0) + \int_0^t dt' \frac{r^2(t)}{r^2(t')}b(t') \tag{26}$$

where we have defined the quantity

$$r(t) = \exp\left(-\int_0^t dt' a(t')\right). \tag{27}$$

For the global two-time correlation (13), the sum of Eqs. (15) and (17) gives $(\partial/\partial t)C_{\mathrm{g}}(t,t_{\mathrm{w}})=-aC_{\mathrm{g}}(t,t_{\mathrm{w}})$ and after integration

$$C_{\mathrm{g}}(t,t_{\mathrm{w}}) = \frac{r(t)}{r(t_{\mathrm{w}})}C_{\mathrm{g}}(t_{\mathrm{w}},t_{\mathrm{w}}). \tag{28}$$

To compute the corresponding global response (21), we add Eqs. (22) and (23) to obtain

$$\frac{\partial}{\partial t}R_{\mathrm{g}}(t,t_{\mathrm{w}}) = - aR_{\mathrm{g}}(t,t_{\mathrm{w}}).$$

The solution for the global response is then given by

$$R_{\mathrm{g}}(t,t_{\mathrm{w}}) = \frac{r(t)}{r(t_{\mathrm{w}})}\beta\{1-\tanh^2[\beta m(t_{\mathrm{w}})]\} \tag{29}$$

where we have used $R_{\mathrm{g}}(t,t)=R_{\mathrm{loc}}(t,t)$ and Eq. (24).

By combining the last two results we get an exact analytical expression for the global FDR

$$X_{\mathrm{g}}(t,t_{\mathrm{w}}) = \frac{TR_{\mathrm{g}}(t,t_{\mathrm{w}})}{\partial C_{\mathrm{g}}(t,t_{\mathrm{w}})/\partial t_{\mathrm{w}}} = \frac{T[1-a(t_{\mathrm{w}})]}{b(t_{\mathrm{w}}) - a(t_{\mathrm{w}})C_{\mathrm{g}}(t_{\mathrm{w}},t_{\mathrm{w}})} \tag{30}$$

where we have also used the fact that $\partial r(t_{\mathrm{w}})/\partial t_{\mathrm{w}}=-a(t_{\mathrm{w}})r(t_{\mathrm{w}})$. Equation (30) shows explicitly that the global FDR depends only on the earlier time t_{w}, which is a feature often seen in simple mean-field models. From the general expression (30) we can now analyze the asymptotic FDR for different initial conditions.

1. Zero initial magnetization

We consider first the standard case where the system is initially unmagnetized. This implies $a=1-\beta$, $b=2$, and $r(t)=\exp(-at)$. From Eq. (26) the equal-time correlation is given by

$$C_{\mathrm{g}}(t,t) = e^{-2at}C_{\mathrm{g}}(0,0) + (1-e^{-2at})(b/2a). \tag{31}$$

For high temperatures $T>T_{\mathrm{c}}=1$, where $a>0$, this converges exponentially to its equilibrium value $b/2a=1/a$; the FDR approaches the limiting value $X_{\mathrm{g}}^{\infty}=T(1-a)/(2-a/a)=1$ and the system equilibrates as expected. Below the critical temperature, on the other hand, a is negative and $C_{\mathrm{g}}(t,t)$ diverges exponentially so that $X_{\mathrm{g}}^{\infty}=0$. At criticality, finally, where $T=1$ and $a=0$, the equal-time correlator grows linearly as $C_{\mathrm{g}}(t,t)=C_{\mathrm{g}}(0,0)+2t$ and the FDR has a nontrivial finite limit $X_{\mathrm{g}}^{\infty}=1/2$. These results can be summarized as follows:

$$X_{\mathrm{g}}^{\infty} = \begin{cases} 0, & T<T_c, \\ 1/2, & T=T_c, \\ 1, & T>T_c. \end{cases} \tag{32}$$

These FDR values for the long-range Ising ferromagnet with zero initial magnetization are identical to those obtained for finite-dimensional spherical ferromagnets above their upper critical dimension [13,26], as one might have expected on physical grounds.

2. Nonzero initial magnetization

For nonzero initial values of the magnetization, which without loss of generality we take as positive, one again needs to distinguish temperatures above, below and at the critical temperature. In the first two cases, Eq. (10) tells us that $m(t)$ decays exponentially to its equilibrium value m_{eq}. This value is $m_{\mathrm{eq}}=0$ for $T>T_{\mathrm{c}}$, while for $T<T_{\mathrm{c}}$ it is the (positive) solution of $m_{\mathrm{eq}}=\mathrm{th}(m_{\mathrm{eq}})=\tanh(\beta m_{\mathrm{eq}})$. Along with m, the quantities a and b also converge quickly to $a_{\mathrm{eq}}=1$

$-\beta(1-m_{\rm eq}^2)$ and $b_{\rm eq}=2(1-m_{\rm eq}^2)$. As $a_{\rm eq}$ is just the relaxation rate of $m(t)$ to $m_{\rm eq}$, it is positive both above and below the critical temperature. From Eq. (31) the equal-time correlator then tends to $b_{\rm eq}/(2a_{\rm eq})$ and so the FDR approaches the value [32]

$$X_{\rm g}^\infty = \frac{1-m_{\rm eq}^2}{b_{\rm eq}-b_{\rm eq}/2} = 1. \tag{33}$$

This is as expected since the system reaches equilibrium exponentially quickly.

The interesting case is a quench to criticality ($T_c=1$) with nonzero initial magnetization $m(0)$. Here Eq. (10) for the magnetization yields the asymptotic power-law decay

$$m(t) = \sqrt{3/2t} \tag{34}$$

independently of initial conditions. Also $a(t)=\tanh^2[m(t)]$ which at long times becomes $a(t)=3/(2t)$; as a consequence, the function $r(t)$ from Eq. (27) scales asymptotically as $\exp[-(3/2)\ln t]=t^{-3/2}$. For the equal-time correlation (26), both the term including the initial condition $C_{\rm g}(0,0)$ and the correction arising from the approach of $b(t)=2\{1-m(t)\tanh[m(t)]\}=2-3/t+\cdots$ to its limit value are then subleading and one has at long times $C_{\rm g}(t,t)=2\int_0^t dt'(t/t')^{-3}=t/2$. The product $a(t)C_{\rm g}(t,t)$ thus approaches $[3/(2t)]\times(t/2)=3/4$ and the global FDR (30) tends to

$$X_{\rm g}^\infty = \frac{1}{2-3/4} = \frac{4}{5}.$$

This result can also be obtained directly from the long-time forms of the correlation (28) and response (29),

$$C_{\rm g}(t,t_{\rm w}) = \frac{t_{\rm w}}{2}\left(\frac{t}{t_{\rm w}}\right)^{-3/2}, \quad R_{\rm g}(t,t_{\rm w}) = \left(\frac{t}{t_{\rm w}}\right)^{-3/2}.$$

Again, we can summarize the results:

$$X_{\rm g}^\infty = \begin{cases} 1, & T<T_c, \\ 4/5, & T=T_c, \\ 1, & T>T_c. \end{cases} \tag{35}$$

The difference between these FDR values and those for the unmagnetized case, Eq. (32), is a clear signature of the difference in the underlying coarsening dynamics. For $T<T_c$ it is physically obvious that the processes involved are very different: in the magnetized case the system equilibrates exponentially quickly, whereas for $m=0$ it ages indefinitely and equilibrium is never established.

The result that also the FDR *at criticality* depends on whether the system is initially magnetized or not, on the other hand, is highly nontrivial. Indeed, one might have expected that the difference between the two cases becomes negligible at long times because the magnetization decays toward zero even in the initially magnetized scenario. Our explicit results show that this relaxation of $m(t)$ does contribute significantly to the FDR, which acquires a nontrivial nonzero limiting value. The latter is distinct from the standard results $X^\infty=1/2$, indicating that coarsening in the presence of a nonzero magnetization belongs to a different dynamical universality class from coarsening at $m=0$.

B. Local FDR

Let us now check if the results obtained for global quantities can be reproduced from the local FDR. This is important because the concept of a nonequilibrium temperature is based on its independence on the choice of observable, and also because numerical work has often focused on the simulation of local correlation and response functions [13,33,34].

The equal-time values of the local correlation and response are given in Eqs. (16) and (24). From Eqs. (17) and (23) the corresponding two-time quantities decay exponentially, so that

$$C_{\rm loc}(t,t_{\rm w}) = [1-m^2(t_{\rm w})]e^{-\tau}, \tag{36}$$

$$R_{\rm loc}(t,t_{\rm w}) = \beta\{1-\tanh^2[\beta m(t_{\rm w})]\}e^{-\tau} \tag{37}$$

where $\tau=t-t_{\rm w}$. The FDR corresponding to the local correlation and response follows as

$$X_{\rm loc}(t,t_{\rm w}) = \frac{1-\tanh^2[\beta m(t_{\rm w})]}{1-m^2(t_{\rm w})-2m(t_{\rm w})\partial m(t_{\rm w})/\partial t_{\rm w}} \tag{38}$$

and again only depends on the earlier time $t_{\rm w}$. If the system starts in an unmagnetized state then $m=0$ at all times and therefore $X_{\rm loc}^\infty=1$ for all temperatures. For nonzero initial values of the magnetization and $T>T_{\rm c}$, $m(t)$ decays exponentially to zero and so $X_{\rm loc}(t_{\rm w}\to\infty)\to 1$. For $T<T_{\rm c}$, $m(t)$ also decays exponentially, but to a nonzero equilibrium value $m_{\rm eq}$. Nevertheless, because $m_{\rm eq}=\tanh(\beta m_{\rm eq})$, Eq. (38) implies that again $X_{\rm loc}(t_{\rm w}\to\infty)\to 1$. At criticality, finally, inserting $m_0^2(t)=3/(2t)$ into Eq. (38) shows that also here $X_{\rm loc}(t_{\rm w}\to\infty)\to 1$, though the convergence is now as a power law ($\sim 1/t_{\rm w}^2$) rather than exponentially. Therefore

$$X_{\rm loc}^\infty = 1 \quad \text{for all } T. \tag{39}$$

In summary, the limiting FDR obtained from the local correlation and response does not pick up any signature of the phase transition at $T_{\rm c}=1$, whether the system is magnetized or not. We will see in Sec. V that this is a somewhat pathological consequence of taking $N\to\infty$ before looking at long times, and that finite-N corrections restore the expected correspondence between local and global measurements.

IV. FINITE-DIMENSIONAL MODELS FOR LARGE d

One would expect that the behavior observed above for the long-range model should also appear in short-range models above their critical dimension. We therefore now extend our discussion to the Ising model on a d-dimensional hypercubic lattice with nearest neighbor (NN) interactions, in the limit of large d. A complication in this case is that there are multiple scalings of the spin correlation functions. For example, local correlations are $O(1)$, those between n.n. spins scale as $O(1/d)$, and those between next nearest neighbours (NNN) as $O(1/d^2)$. In order to capture the $O(1)$ contribution of these correlation functions (and the corresponding responses) it is useful to consider the Fourier transforms

$$C_{\mathbf{q}}(t,t_{\mathrm{w}}) = \sum_{l} e^{i\mathbf{q}\cdot(\mathbf{r}_l-\mathbf{r}_j)} C_{lj}(t,t_{\mathrm{w}}),$$

$$R_{\mathbf{q}}(t,t_{\mathrm{w}}) = \sum_{l} e^{i\mathbf{q}\cdot(\mathbf{r}_l-\mathbf{r}_j)} R_{lj}(t,t_{\mathrm{w}}).$$

For example, the $2d$ NN spins with their correlations $C_{i,j}(t,t_{\mathrm{w}})$ of $O(1/d)$ given an overall contribution to $C_{\mathbf{q}}(t,t_{\mathrm{w}})$ of $O(1)$; the same is true for the $O(d^2)$ NNN spins with their $O(1/d^2)$ correlations and so on.

The Hamiltonian of the short-range model is, by analogy with Eq. (1),

$$H = -\frac{1}{2d}\sum_{(i,j)} \sigma_i\sigma_j - \sum_i h_i^{\mathrm{ext}}\sigma_i.$$

In the interaction term the sum now runs over all NN pairs of spins; the interaction strength has been chosen to get the same critical temperature, $T_c=1$, in the limit $d\to\infty$ as in the long-range model. The local fields are now given by $h_i = h_i^{\mathrm{ext}} + (2d)^{-1}\Sigma_k\sigma_k$ instead of Eq. (2), with the sum running over all NNs of i. For large d the field fluctuations Δh_i are small, of $O(d^{-1/2})$, and so one can again linearize in Δh_i.

As for the long-range model, we proceed to study the correlation and response functions in this model. The general equations (7) and (8) can be used to derive the dynamical equations for the correlations. To arrive at explicit expressions, we need to analyze the correlations between spins and local fields. They can be expressed as

$$\langle \Delta h_i \Delta\sigma_j\rangle = \frac{1}{2d}\sum_k \langle\Delta\sigma_k\Delta\sigma_j\rangle = \frac{1}{2d}\sum_k C_{kj}$$

with Fourier transform

$$\sum_l e^{i\mathbf{q}\cdot(\mathbf{r}_l-\mathbf{r}_j)}\langle\Delta h_l\Delta\sigma_j\rangle = (1-\omega_{\mathbf{q}})C_{\mathbf{q}}.$$

Here

$$\omega_{\mathbf{q}} = 1 - \frac{1}{d}\sum_{a=1}^{d}\cos q_a$$

and the q_a are the spatial components of the wave vector $\mathbf{q}$. Using the large-d expansion of the free energy [35] one can show that equilibrium correlations involving higher powers of the field fluctuation, e.g., $\langle(\Delta h_i)^2\Delta\sigma_j\rangle$, have Fourier transforms which are suppressed by $O(1/d)$ (away from criticality). Following the same reasoning as for the long-range ferromagnet, we discard these subleading contributions. For the magnetization, this leads back to the expected mean-field equation of motion (10). The evolution of the equal-time correlations follows by linearization and Fourier transformation of (7) as

$$\frac{\partial}{\partial t}C_{\mathbf{q}}(t,t) = -2C_{\mathbf{q}}(t,t) + 2\mathrm{th}'(m)(1-\omega_{\mathbf{q}})C_{\mathbf{q}}(t,t) + b(t) \tag{40}$$

where the last term accounts for the fact that the local correlations $C_{ii}(t,t)=1-m^2(t)$ have a different equation of motion from the nonlocal ones. One can write an explicit expression for $b(t)$ but this is not helpful because it depends itself on the $C_{\mathbf{q}}(t,t)$ which we are trying to find. Instead we first solve Eq. (40) for arbitrary $b(t)$ and then determine the latter such that the local correlations come out correctly. Generalizing the definitions of a, Eq. (18), and r, Eq. (27), to

$$a_{\mathbf{q}} = 1 - \mathrm{th}'(m)(1-\omega_{\mathbf{q}}),$$

$$r_{\mathbf{q}}(t) = \exp\left(-\int_0^t dt'\, a_{\mathbf{q}}(t')\right),$$

the solution of Eq. (40) reads

$$C_{\mathbf{q}}(t,t) = r_{\mathbf{q}}^2(t)C_{\mathbf{q}}(0,0) + \int_0^t dt'\,\frac{r_{\mathbf{q}}^2(t)}{r_{\mathbf{q}}^2(t')}b(t'). \tag{41}$$

The function $b(t')$ can now be determined from the constraint that $\int(dq)C_{\mathbf{q}}(t,t)=C_{ii}(t,t)$, where the shorthand notation $\int(dq)$ indicates the integral over $\mathbf{q}\in[-\pi,\pi]^d$ normalized by $(2\pi)^d$. Integrating Eq. (41) we find that

$$1-m^2(t) = \int (dq) r_{\mathbf{q}}^2(t)C_{\mathbf{q}}(0,0) + \int_0^t dt' \int (dq)\frac{r_{\mathbf{q}}^2(t)}{r_{\mathbf{q}}^2(t')}b(t'). \tag{42}$$

For simplicity, we focus in the following on initially uncorrelated spins, i.e., $C_{\mathbf{q}}(0,0)=1-m^2(0)$. Using that

$$\int (dq) e^{x(1-\omega_{\mathbf{q}})} = \left(\int_{-\pi}^{\pi}\frac{dq_1}{2\pi}e^{(x/d)\cos q_1}\right)^d = \left(1+\frac{x^2}{4d^2}+\cdots\right)^d = 1 + O\left(\frac{1}{d}\right)$$

Eq. (42) then simplifies for large d to

$$1-m^2(t) = e^{-2t}[1-m^2(0)] + \int_0^t dt'\, e^{-2(t-t')}b(t').$$

For $m(t)=0$ at all times this gives the constant value $b(t)=2$. For more general scenarios, b still converges to this value at long times as long as the magnetization decays to zero, i.e., for $T\geqslant T_{\mathrm{c}}$.

For the two-time correlations, linearization of (8) in Δh_i yields the evolution equation $(\partial/\partial t)C_{\mathbf{q}}(t,t_{\mathrm{w}})=-a_{\mathbf{q}}C_{\mathbf{q}}(t,t_{\mathrm{w}})$ and thus

$$C_{\mathbf{q}}(t,t_{\mathrm{w}}) = \frac{r_{\mathbf{q}}(t)}{r_{\mathbf{q}}(t_{\mathrm{w}})}C_{\mathbf{q}}(t_{\mathrm{w}},t_{\mathrm{w}}).$$

The instantaneous response remains purely local as in Eq. (A12), with $R_{\mathbf{q}}(t,t)=\mathrm{th}'(m)$ to leading order in $1/d$. The evolution of the two-time response is obtained as in the long-range case, by linearizing Eq. (A8) in the applied field. This gives simply $(\partial/\partial t)R_{\mathbf{q}}(t,t_{\mathrm{w}})=-a_{\mathbf{q}}R_{\mathbf{q}}(t,t_{\mathrm{w}})$, which integrates to

$$R_{\mathbf{q}}(t,t_{\mathrm{w}})=\frac{r_{\mathbf{q}}(t)}{r_{\mathbf{q}}(t_{\mathrm{w}})}\mathrm{th}'[m(t_{\mathrm{w}})].$$

On the basis of the correlation and response functions one can define a Fourier-component FDR. By analogy with Eq. (30) this can be simplified to

$$X_{\mathbf{q}}(t,t_{\mathrm{w}})=\frac{TR_{\mathbf{q}}(t,t_{\mathrm{w}})}{(\partial C_{\mathbf{q}}/\partial t_{\mathrm{w}})(t,t_{\mathrm{w}})}=\frac{\mathrm{th}'[m(t_{\mathrm{w}})]}{b(t_{\mathrm{w}})-a_{\mathbf{q}}(t_{\mathrm{w}})C_{\mathbf{q}}(t_{\mathrm{w}},t_{\mathrm{w}})}. \tag{43}$$

The FDR for the global magnetization, which is the Fourier component of the spins with $\mathbf{q}=\mathbf{0}$, is obtained from this expression as a special case. Let us now concentrate on quenches to T_c where we found for the long-range model a nontrivial dependence of the results on the initial conditions.

A. Zero initial magnetization

For zero initial magnetization, $m(t)=0$ at all times and so $a_{\mathbf{q}}(t)=\omega_{\mathbf{q}}$ and $r_{\mathbf{q}}(t)=\exp(-\omega_{\mathbf{q}}t)$. Correspondingly, the equal-time correlations (41) simplify to

$$C_{\mathbf{q}}(t_{\mathrm{w}},t_{\mathrm{w}})=\exp(-2\omega_{\mathbf{q}}t_{\mathrm{w}})+\frac{1}{\omega_{\mathbf{q}}}(1-e^{-2\omega_{\mathbf{q}}t_{\mathrm{w}}}). \tag{44}$$

The relevant scaling variable is clearly $w\equiv\omega_{\mathbf{q}}t_{\mathrm{w}}$, so we will focus on the limit of long times and low "frequencies" $\omega_{\mathbf{q}}$ taken such that that w remains constant. Since $\omega_{\mathbf{q}}=\mathbf{q}^2/(2d)$ for small $\omega_{\mathbf{q}}$, the scaling $\omega_{\mathbf{q}}\sim t_{\mathrm{w}}^{-1}$ reflects the growing length scale $1/q\sim t_{\mathrm{w}}^{1/2}$ of the correlations as the system coarsens. The first term in Eq. (44), which arises from the decay of initial correlations, then becomes negligible compared to the second and we get

$$X_{\mathbf{q}}(t,t_{\mathrm{w}})\simeq\frac{1}{2-\omega_{\mathbf{q}}[1-\exp(-2\omega_{\mathbf{q}}t_{\mathrm{w}})]/\omega_{\mathbf{q}}} \tag{45}$$

$$=\frac{1}{1+\exp(-2\omega_{\mathbf{q}}t_{\mathrm{w}})}. \tag{46}$$

For length scales short compared to the time-dependent correlation length, where $\omega_{\mathbf{q}}t_{\mathrm{w}}\gg 1$, $X_{\mathbf{q}}$ becomes equal to unity as expected because of effective equilibration on such short scales. For much larger length scales ($\omega_{\mathbf{q}}t_{\mathrm{w}}\ll 1$), and in particular in the limit $\omega_{\mathbf{q}}\to 0$ which gives the FDR for the magnetization, $X_{\mathbf{q}}$ approaches 1/2. These two limits can be seen in Fig. 1 and are consistent with the results for the long-range ferromagnet, but here we see that in addition one can interpolate smoothly between the two limits by varying the length scale considered. Similar behavior is also found in the Ising chain [14], i.e., for $d=1$, and in the spherical model [26]. Given that our calculation is based on a linearization in the local field fluctuations, it is not surprising that the result (46) can also be obtained from a Gaussian field theory [36].

B. Nonzero initial magnetization

For nonzero initial magnetization and at criticality, the magnetization again decays according to Eq. (34). One then

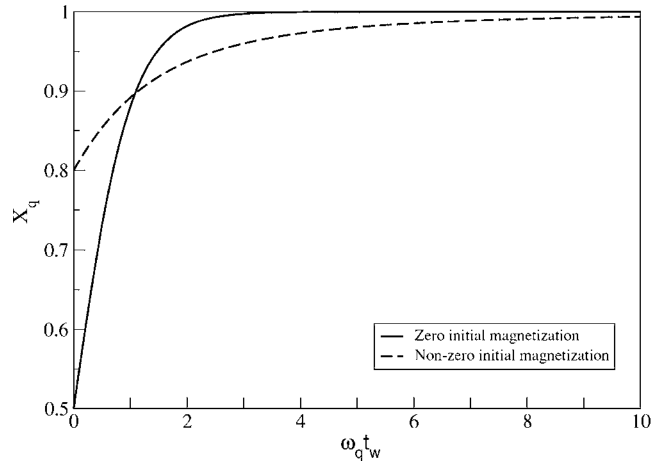

FIG. 1. Dependence of the FDR on the length scale in the large-d short-range ferromagnet, for initial conditions with zero [solid line, Eq. (46)] and nonzero [dashed line, Eq. (47)] magnetization. Shown on the x axis is the scaling variable $\omega_{\mathbf{q}}t_{\mathrm{w}}\sim q^2t_{\mathrm{w}}$, which is proportional to the squared ratio of the time-dependent correlation length ($\sim t_{\mathrm{w}}^{1/2}$) to the length scale being probed by the chosen observable ($1/q$).

has $\mathrm{th}'[m(t_{\mathrm{w}})]=1-3/(2t_{\mathrm{w}})$ to leading order for large times, and consequently $a_{\mathbf{q}}(t_{\mathrm{w}})=1-[1-3/(2t_{\mathrm{w}})](1-\omega_{\mathbf{q}})$. This gives

$$r_{\mathbf{q}}(t)=\exp\left(-\int_0^t dt'\,a_{\mathbf{q}}(t')\right)=e^{-t+[t-(3/2)\ln t](1-\omega_{\mathbf{q}})}$$
$$=t^{-(3/2)(1-\omega_{\mathbf{q}})}e^{-\omega_{\mathbf{q}}t}$$

and for long times, the equal-time correlations (41) follow as

$$C_{\mathbf{q}}(t_{\mathrm{w}},t_{\mathrm{w}})=t_{\mathrm{w}}^{-3(1-\omega_{\mathbf{q}})}e^{-2\omega_{\mathbf{q}}t_{\mathrm{w}}}+\int_0^{t_{\mathrm{w}}}dt'\left(\frac{t'}{t_{\mathrm{w}}}\right)^{3(1-\omega_{\mathbf{q}})}e^{-2\omega_{\mathbf{q}}(t_{\mathrm{w}}-t')}b(t').$$

For $t_{\mathrm{w}}\to\infty$ at fixed $w=\omega_{\mathbf{q}}t_{\mathrm{w}}$, the first term and the approach of $b(t')$ to its limit $b=2$ give only subleading corrections and one gets asymptotically

$$C_{\mathbf{q}}(t_{\mathrm{w}},t_{\mathrm{w}})=\frac{1}{\omega_{\mathbf{q}}}\mathcal{F}_C(\omega_{\mathbf{q}}t_{\mathrm{w}}),$$

$$\mathcal{F}_C(w)=2w\int_0^1 dy\,y^3e^{-2w(1-y)}.$$

The Fourier-component FDR (43) therefore becomes

$$X_{\mathbf{q}}^{-1}(t,t_{\mathrm{w}})=2-\{1-[1-3/(2t_{\mathrm{w}})](1-\omega_{\mathbf{q}})\}\omega_{\mathbf{q}}^{-1}\mathcal{F}_C(\omega_{\mathbf{q}}t_{\mathrm{w}})$$
$$=2-[3/(2w)+1]\mathcal{F}_C(w) \tag{47}$$

where we have neglected all terms that are subleading ($\sim 1/t_{\mathrm{w}}$) for long times. For short length scales, $w\gg 1$, one sees easily that $\mathcal{F}_C\to 1$ and so $X_{\mathbf{q}}\to 1$, demonstrating the expected equilibration. On the other hand, for length scales much larger than the time-dependent correlation length, i.e., $w\ll 1$, one has $\mathcal{F}_C=w/2$ and so $X_{\mathbf{q}}\to(2-3/4)^{-1}=4/5$. This applies in particular to the FDR for the magnetization ($\mathbf{q}$

$=\mathbf{0}$) and so is consistent with the result found above for the long-range model. Again there is a smooth length-scale dependence of the FDR that interpolates between local equilibrium and the nontrivial FDR, $X^\infty=4/5$, for large length scales; see the dashed line in Fig. 1. This is rather reassuring: it tells us that there is nothing special about the magnetization, i.e., $\mathbf{q}=\mathbf{0}$, even though in a magnetized system this is the only Fourier component that has a nonzero average. In physical terms, the FDR for the magnetization can also be observed by looking at length scales that are much smaller than the system size, as long as they are large compared to the time-dependent correlation length.

We have only discussed the FDR for pure Fourier components here. However, if one considers more general correlations, say of spins across some finite range, one simply gets a mixture of the Fourier-component FDRs. The resulting $X(t,t_\mathrm{w})$ will be a mixture of all $X_\mathbf{q}(t,t_\mathrm{w})$ and interpolate between $X(t_\mathrm{w},t_\mathrm{w})=1$ at equal times and a nontrivial asymptotic value X^∞ in the limit $t\gg t_\mathrm{w}$ of well-separated times. Following the reasoning in the one-dimensional case [14], one can show that this asymptotic FDR is always identical to the FDR for the longest wavelength, i.e., $X_{\mathbf{q}=\mathbf{0}}$, because the limit $t\gg t_\mathrm{w}$ suppresses the contributions from all nonzero wave vectors. In particular, this means that *all* observables that are linear in the spins, including local correlation and response, will give $X^\infty=1/2$ for critical coarsening at zero magnetization, and $X^\infty=4/5$ for the magnetized case. (Similarly to the long-range case discussed in the next section one can show that these asymptotic FDR values would only be observed for rather long time differences $t-t_\mathrm{w}$, of order 2^d or larger, while for shorter time differences apparent equilibrium behavior is obtained.)

V. FINITE-SIZE CORRECTIONS FOR THE LONG-RANGE FERROMAGNET

In the short-range model just discussed, local and global observables give the same limiting FDR X^∞. But in the long-range model of Sec. III, this correspondence appears to be broken because the local correlation and response functions do not pick up any nonequilibrium effects. To analyze the origin of this discrepancy, we now study the $1/N$ corrections to C_loc [Eq. (11)] and R_loc [Eq. (19)]. The calculations are sketched in Appendix B and we only give the main results here.

Let us first consider again the unmagnetized case at high temperatures, $T>T_\mathrm{c}$. Keeping terms up to $O(N^{-1})$, we find for the local correlation

$$\tilde{C}_\mathrm{loc}(t,t_\mathrm{w})=e^{-\tau}\left(1-\frac{(1-\beta)^{-1}+\beta\tau}{N}\right)+\frac{e^{-(1-\beta)\tau}}{N(1-\beta)} \qquad (48)$$

where $\tau=t-t_\mathrm{w}$ as before. The second term becomes dominant over the first for time differences $\tau\approx T\ln N$, i.e., on a time scale that grows only logarithmically with the system size. (The $1/N$ corrections in the large brackets can always be neglected; they would become relevant only for $\tau\sim N$, but by then the second term is exponentially larger than the first.) For a finite-size system the correction term therefore dominates the long-time dynamics, causing the decay rate of the correlation to slow from 1 to $1-\beta$ at $\tau\approx T\ln N$. The corresponding local response is related by the FDT to the correlation; this is as expected because for the long times considered here the system is in equilibrium. A similar argument indicates that the above long-time results for $T>T_\mathrm{c}$ remain unchanged if the initial magnetization is nonzero.

Next we analyze the nonequilibrium dynamics at criticality $(T=T_c=1)$ starting from zero magnetization; here we would hope to retrieve from the $O(1/N)$ correction terms a long-time FDR of $X_\mathrm{loc}^\infty=1/2$. One finds

$$\tilde{C}_\mathrm{loc}(t,t_\mathrm{w})=e^{-\tau}+\frac{1}{N}[C_\mathrm{g}(t_\mathrm{w},t_\mathrm{w})(1-e^{-\tau})-\tau e^{-\tau}]. \qquad (49)$$

Using the fact that $C_\mathrm{g}(t_\mathrm{w},t_\mathrm{w})=2t_\mathrm{w}+C_\mathrm{g}(0,0)$ at criticality, the $1/N$ expansion now breaks down for $t_\mathrm{w}\sim N$, where the correction term becomes $O(1)$ rather than $O(N^{-1})$ as the expansion assumes. For smaller values of t_w the expansion remains valid, however, and the t_w derivative of the correlation becomes

$$\frac{\partial}{\partial t_\mathrm{w}}\tilde{C}_\mathrm{loc}(t,t_\mathrm{w})=e^{-\tau}+\frac{2}{N}(1-e^{-\tau})-\frac{1}{N}e^{-\tau}(C_\mathrm{g}(t_\mathrm{w},t_\mathrm{w})+1+\tau).$$

The third term can be neglected in the reliable regime $t_\mathrm{w}\ll N$, but the second one again becomes dominant over the leading contribution for $\tau\approx\ln N$; for $\tau-\ln N\gg 1$, one then has $(\partial/\partial t_\mathrm{w})C_\mathrm{loc}(t,t_\mathrm{w})=2/N$. The corresponding local response reads

$$\tilde{R}_\mathrm{loc}(t,t_\mathrm{w})=e^{-\tau}\left(1-\frac{1}{N}[C_\mathrm{g}(t_\mathrm{w},t_\mathrm{w})+1+\tau]\right)+\frac{1}{N}.$$

The second term becomes dominant over the first for $\tau\approx\ln N$, while the $1/N$ corrections in the first term can always be neglected for $t_\mathrm{w}\ll N$. For $\tau-\ln N\gg 1$ the corrected FDR is therefore $\tilde{X}_\mathrm{loc}(t,t_\mathrm{w})=(1/N)/(2/N)=1/2$. To summarize, for systems that are old $(t_\mathrm{w}\gg 1)$ but not yet equilibrated $(t_\mathrm{w}\ll N)$, the FDR as a function of the time difference τ crosses over from the equilibrium value $\tilde{X}_\mathrm{loc}=1$ to the nonequilibrium value $\tilde{X}_\mathrm{loc}^\infty=1/2$ on a time scale $\tau\approx\ln N$. In any finite-size system the limiting value of the *local* FDR therefore agrees with the *global* one, just as the local-global correspondence leads one to expect. If the limit $N\to\infty$ is taken before the long-time limit, as we did in Sec. II, then one implicitly discards the nonequilibrium regime. This is what leads to the apparent breaking of the correspondence with the global results.

We now compare with the corresponding results for the nonequilibrium dynamics at criticality starting from nonzero initial magnetization. Evaluating the $1/N$ correction terms in the regime $\tau\gg 1$ where they are potentially relevant, we find

$$\tilde{C}_\mathrm{loc}(t,t_\mathrm{w})=e^{-\tau}+\frac{t_\mathrm{w}}{2N}\left(\frac{t}{t_\mathrm{w}}\right)^{-3/2}, \qquad (50)$$

$$\tilde{R}_{\mathrm{loc}}(t,t_{\mathrm{w}}) = e^{-\tau} + \frac{1}{N}\left(\frac{t}{t_{\mathrm{w}}}\right)^{-3/2}. \tag{51}$$

As in the unmagnetized case, the correction terms become significant for $\tau \approx \ln N$. For larger time differences the local quantities become proportional to the global ones; as a consequence, the corrected local FDR $\tilde{X}_{\mathrm{loc}}$ crosses over from 1 to the global FDR $\tilde{X}^{\infty}_{\mathrm{loc}} = X^{\infty}_{\mathrm{g}} = 4/5$. Our main conclusion of this section is, therefore, that in any finite-size system the local-global correspondence is preserved.

We note as an aside that the relevant scaling variable for the above crossovers in $\tilde{X}_{\mathrm{loc}}$ is $N\exp(-\tau)$. Plots of $\tilde{X}_{\mathrm{loc}}$ versus $N\exp(-\tau)$ would look similar to those in Fig. 1 for both the magnetized and unmagnetized cases, though one has to bear in mind that somewhat different quantities are being plotted: Fig. 1 refers to the length-scale dependence of the limiting FDR, whereas here we have a fixed short length scale (local correlation and response) and are looking at the system-size-dependent crossover in time of the FDR to its limiting value.

We have not considered the $1/N$ corrections at $T < T_{\mathrm{c}}$ in this section because it turns out—consistent with the fast equilibration when starting from nonzero magnetization—that here the $1/N$ expansion breaks down already for system ages t_{w} of order $\ln N$.

VI. SUMMARY AND DISCUSSION

In this paper we have solved analytically the nonequilibrium dynamics of the long-range Ising ferromagnet with Glauber dynamics, initially in the thermodynamic limit and then including also the leading finite-size corrections; our focus was on the correlation and response functions and the associated fluctuation-dissipation ratio. We have also analyzed the corresponding short-range model in the limit of large dimension, which provides useful additional insights into the length-scale dependence of the FDR.

Our main result is that different nontrivial values of the limiting FDR X^{∞} can result depending on the initial conditions. In particular, for quenches to the critical temperature we find that $X^{\infty} = 1/2$ in the standard scenario where the system is initially unmagnetized, while $X^{\infty} = 4/5$ if the initial magnetization is nonzero. We are not aware of any previous observations of a nontrivial (nonzero) value of X^{∞} arising from initial conditions other than those traditionally considered; earlier studies [28,29] of strongly correlated initial conditions had always found either the standard value or $X^{\infty} = 0$.

Our findings show that critical coarsening processes are fundamentally different depending on whether the system is magnetized or not, and the two cases must be considered as belonging to different dynamical universality classes. One would have certainly expected such a distinction *below* T_c, where in the magnetized case the system equilibrates rapidly. Our finding that the difference persists even *at* T_c is much less obvious, seeing as even in the initially magnetized case the magnetization does decay toward zero at long times. The limiting FDR thus turns out to be a useful probe for distinguishing different classes of nonequilibrium dynamics. Of course, the differences in the FDR also imply that the associated effective temperatures T/X^{∞} differ between the magnetized and unmagnetized cases; this emphasises that the same system can have different nonequilibrium effective temperatures depending on its initial preparation, which then reflect the physical differences in the ensuing nonequilibrium dynamics.

One may ask whether our results are peculiar to the relatively simple scenarios that we have considered. This is not so: calculations in the spherical ferromagnet quenched to criticality [26] give exactly the same limiting FDR $X^{\infty} = 4/5$ for the magnetized case, in all dimensions $d > 4$. To understand where this particular value comes from, one can write down a phenomenological Langevin equation for the evolution of the fluctuating magnetization,

$$\frac{dm}{dt} = -m^n + h + \xi, \tag{52}$$

where h is an external magnetic field and ξ is white noise with variance $O(1/N)$. In the case $m = 0$, because fluctuations around this value will be of $O(N^{-1/2})$, the nonlinear term on the right-hand side of Eq. (52) can be neglected. One thus recovers simple diffusive dynamics independently of n (as long as $n > 1$) which results in the familiar value $X^{\infty} = 1/2$ for the limiting FDR. On the other hand, if the magnetization is initially nonzero then Eq. (52) predicts that its average value decays as $\langle m\rangle \sim t^{-1/(n-1)}$ when there is no external field, whereas for $h \neq 0$ it approaches $\langle m\rangle \sim h^{1/n}$. Comparing with the standard scalings $\langle m\rangle \sim t^{-\beta/(\nu z)}$ and $\langle m\rangle \sim h^{1/\delta}$, respectively, one sees that we require $n = \delta$ and $1/(n-1) = \beta/(\nu z)$; these two choices for n are consistent with each other only if the mean-field relation $z = 2 - \eta$ holds. We can then go ahead and, by linearizing Eq. (52) in the small deviations of m from its average $\langle m\rangle$, calculate its correlation and response function. This simple calculation gives for the limiting FDR $X^{\infty} = (3n-1)/(4n-2) = (2\beta + 3\nu z)/(2\beta + 4\nu z)$. Inserting the mean-field exponents $z = 2$, $\beta = 1/2$, and $\nu = 1/2$, which imply $n = 3$, then leads to $X^{\infty} = 4/5$.

Summarizing, the phenomenological Langevin equation (52) predicts two different results for the limiting FDR of mean-field ferromagnets, depending on the initial condition:

$$X^{\infty}_{\mathrm{g}} = \begin{cases} 1/2 & \text{if } m = 0, \\ 4/5 & \text{if } m \neq 0. \end{cases}$$

These are precisely the values that we found in our explicit calculations for the long-range and high-dimensional short-range models. One may then wonder whether this phenomenological approach can also be used to predict FDR values below the upper critical dimension. The spherical model in $d < 4$, for example, has non-mean-field exponents but still satisfies $z = 2 - \eta$, so that the Langevin description is at least internally consistent. However, explicit calculations [26] show that the values of the FDR it predicts are incorrect. The Langevin dynamics (52) is therefore too simple to capture the full physics away from mean field. It remains an open question whether appropriately generalized phenomenological descriptions can be used to rationalize FDR values in non-mean-field systems.

We have also investigated in this paper the dependence of the limiting FDR on the observable considered. If the FDR and any associated effective temperature are physically meaningful, one would hope that, e.g., local and global observables would lead to the same limiting FDRs. In the short-range model we showed that this is indeed the case, because the behavior of both types of observables becomes dominated by the slowest, longest-wavelength Fourier modes ($q \approx 0$) in the limit of long times. The FDR for the Fourier modes themselves showed the expected crossover between the values $X_{\mathbf{q}}^{\infty}=1/2$ and $4/5$ for length scales larger than the time-dependent correlation length ($q \ll t_{\mathrm{w}}^{-1/2}$) and $X_{\mathbf{q}}^{\infty}=1$ for shorter, equilibrated, length scales.

In the long-range model, we found that great care is needed when computing local FDRs because the limits $N \to \infty$ and $t-t_{\mathrm{w}} \to \infty$ do not commute. If the thermodynamic limit $N \to \infty$ is taken before the long-time limit (as we did in Sec. II) one gets FDR values that are different from those obtained for global quantities because the nonequilibrium regime is effectively excluded. To find physically meaningful results, one has to take the long-time limit before the thermodynamic one; this then requires that the system size is kept large but finite as in Sec. V. With this, the expected correspondence between local and global FDRs is recovered. These findings are not only of theoretical interest but also have two implications for numerical studies. First, if in long-range models one uses local observables to measure nonequilibrium FDRs, simulations out to very long time differences will be required to obtain meaningful results that can reveal nonequilibrium effects. Second, the fact that for global observables we could go directly to the infinite-system-size limit underlines the general message [14,27] that such global quantities are much more robust tools for detecting FDT violations than their local counterparts.

Our main finding that magnetized and unmagnetized coarsening processes at criticality can belong to different dynamical universality classes clearly deserves wider study in future work. Calculations for the spherical model [26] show that this distinction also holds true below the upper critical dimension $d=4$, and provide explicit (and surprisingly nontrivial) predictions for the resulting FDR values. It would be interesting to complement this with simulation studies of Ising models in $d=2$ and 3. Field-theoretic renormalization-group calculations [16] might also be possible for the $O(n)$ and n-vector models. For $n=1$ these reduce to the Ising universality class, while for $n \to \infty$ one would expect to recover the spherical model results; knowledge of the detailed dependence of the FDR values on the order parameter dimensionality n as one interpolates between these two extreme values should help to round out the physical picture further. After the present work was completed we became aware that a first step in this direction has very recently been taken by the authors of Ref. [37], who calculated the FDR for the n-vector model with a magnetized initial state within an ϵ expansion around $d=2$.

Finally, there is the possibility that there might be yet other initial conditions which give rise to distinct and nontrivial values of the limiting FDR. This does not seem likely, given that the obvious candidate case of strongly correlated but unmagnetized configurations gives $X^{\infty}=0$ and can thus be excluded. Nevertheless, a complete characterization of possible classes of nonequilibrium coarsening induced by different initial conditions certainly remains to be achieved.

ACKNOWLEDGMENTS

A.G. wishes to thank the University of Pompeu Fabra for its support. F.R. is supported by the Ministerio de Educación y Ciencia in Spain (Grant No. BFM2001-3525), STIPCO (Grant No. HPRN-CT-2002-00319), and by the Distinció de la Generalitat de Catalunya. I.P. acknowledges financial support from DGICYT of the Spanish Government and from the Distinció de la Generalitat de Catalunya. All authors have been supported by the SPHINX ESF program.

APPENDIX A: DYNAMICAL EQUATIONS FOR THE LONG-RANGE FERROMAGNET

In this appendix we outline the explicit calculation of the dynamical equations for the correlations and responses. The global correlation function is decomposed to leading order in the system size as

$$\frac{1}{N}\sum_{ij} C_{ij} = [C_{\mathrm{loc}} + O(N^{-1})] + (N-1)\left(\frac{C_{\mathrm{nl}}}{N} + O(N^{-2})\right) = C_{\mathrm{loc}} + C_{\mathrm{nl}} + O(N^{-1}) = C_{\mathrm{g}} + O(N^{-1}) \quad (A1)$$

using Eqs. (11)–(13). At equal times, the local correlations are trivially

$$C_{ii}(t,t) = C_{\mathrm{loc}}(t,t) = 1 - m^2(t) \quad (A2)$$

while for the nonlocal correlations (7) implies to leading order

$$\frac{1}{N}\frac{\partial}{\partial t}C_{\mathrm{nl}}(t,t) = -\frac{2}{N}C_{\mathrm{nl}}(t,t) + \mathrm{th}'(m)(\langle \Delta h_i \Delta\sigma_j\rangle + \langle \Delta\sigma_i \Delta h_j\rangle) = -\frac{2}{N}C_{\mathrm{nl}}(t,t) + \frac{2}{N}\mathrm{th}'(m)C_{\mathrm{g}}(t,t). \quad (A3)$$

In the second line we have used that, for $i \neq j$,

$$\langle \Delta h_i \Delta\sigma_j\rangle = \frac{1}{N-1}\sum_{k\neq i}\langle \Delta\sigma_k \Delta\sigma_j\rangle = \frac{C_{\mathrm{loc}}(t,t) + O(N^{-1})}{N-1} + \frac{N-2}{N-1}\left(\frac{C_{\mathrm{nl}}(t,t)}{N} + O(N^{-2})\right) = \frac{C_{\mathrm{g}}(t,t)}{N} + O(N^{-2}). \quad (A4)$$

Defining the quantity a as in Eq. (18) and bearing in mind that $C_{\mathrm{g}}(t,t)=1-m^2(t)+C_{\mathrm{nl}}(t,t)$, we can then write the evolution equation (A3) for the nonlocal equal-time correlations as

$$\frac{\partial}{\partial t}C_{\mathrm{nl}}(t,t) = -2aC_{\mathrm{nl}}(t,t) + 2\mathrm{th}'(m)(1-m^2).$$

For the two-time correlations we get similarly from Eq. (8)

$$\frac{\partial}{\partial t}C_{ij}(t,t_w) = -C_{ij}(t,t_w) + \mathrm{th}'(m)\langle\Delta h_i(t)\Delta\sigma_j(t_w)\rangle + O(N^{-2}) \tag{A5}$$

which gives for the leading order of the nonlocal terms the dynamical evolution

$$\frac{\partial}{\partial t}C_{nl}(t,t_w) = -C_{nl}(t,t_w) + \mathrm{th}'(m)C_g(t,t_w). \tag{A6}$$

For local correlations, on the other hand, the second term in Eq. (A5) is subleading so that to leading order

$$\frac{\partial}{\partial t}C_{loc}(t,t_w) = -C_{loc}(t,t_w). \tag{A7}$$

Finally we want the dynamical equations for the linear response functions $R_{ij}(t,t_w)=\delta\langle\sigma_i(t)\rangle/\delta h_j^{ext}(t_w)$. We assume that the field is applied to site $j=1$ and, as for the correlation functions, use the appropriate scalings (19) and (20). Setting $h_1^{ext}\equiv h$ and using Eq. (3) we can write

$$\frac{\partial}{\partial t}\langle\sigma_i\rangle = \langle \mathrm{th}(m+\Delta h_i + h\delta\langle h_i\rangle)\rangle - \langle\sigma_i\rangle. \tag{A8}$$

Here m refers to the value of the magnetization for the unperturbed system, while

$$\delta\langle h_i\rangle = \frac{1}{N-1}\sum_{k\neq i} R_{k1}(t,t_w) \tag{A9}$$

is the response function for the average value of the local field at site i. If we expand again in powers of Δh_i, the first nontrivial term is proportional to $\langle(\Delta h_i)^2\rangle$ and does not contribute to leading order. Writing $\langle\sigma_i\rangle=m+hR_{i1}(t,t_w)$, Eq. (A8) thus becomes

$$\frac{\partial m}{\partial t} + h\frac{\partial}{\partial t}R_{i1}(t,t_w) = \mathrm{th}(m+h\delta\langle h_i\rangle) - m - hR_{i1}(t,t_w).$$

Expanding to linear order in h then gives back at $O(h^0)$ the expected dynamical equation (10) for m, while at $O(h)$ one gets

$$\frac{\partial}{\partial t}R_{i1}(t,t_w) = \mathrm{th}'(m)\delta\langle h_i\rangle - R_{i1}(t,t_w).$$

In the nonlocal case ($i\neq 1$), where $\delta\langle h_i\rangle=(R_{loc}+R_{nl})/N+O(N^{-2})$ from Eqs. (A9), (19), and (20), this gives to leading order

$$\frac{\partial}{\partial t}R_{nl}(t,t_w) = \mathrm{th}'(m)[R_{loc}(t,t_w)+R_{nl}(t,t_w)] - R_{nl}(t,t_w). \tag{A10}$$

For the local case $i=1$, on the other hand, the term proportional to $\delta\langle h_i\rangle$ is subleading and one gets simply

$$\frac{\partial}{\partial t}R_{loc}(t,t_w) = -R_{loc}(t,t_w). \tag{A11}$$

These equations can be integrated forward in time once the instantaneous response is known. The latter is purely local, as one sees from

$$R_{ij}(t,t) = \left.\frac{\partial}{\partial h_j^{ext}}\langle \mathrm{th}(h_i^{ext}+m+\Delta h_i) - \sigma_i\rangle\right|_{h^{ext}=0} = \delta_{ij}\langle \mathrm{th}'(m+\Delta h_i)\rangle. \tag{A12}$$

Thus, to leading order

$$R_{loc}(t,t) = \mathrm{th}'(m), \quad R_{nl}(t,t) = 0. \tag{A13}$$

APPENDIX B: 1/N CORRECTIONS FOR THE LONG-RANGE FERROMAGNET

To calculate the $1/N$ corrections to the local correlation and response we expand $C_{ii}=C_{loc,0}+C_{loc,1}/N+O(N^{-2})=\widetilde{C}_{loc}+O(N^{-2})$ and $R_{ii}=R_{loc,0}+R_{loc,1}/N+O(N^{-2})=\widetilde{R}_{loc}+O(N^{-2})$; the magnetization, which enters $C_{loc,1}$, is similarly expanded as $m=m_0+m_1/N+O(N^{-2})$. The quantities $C_{loc,0}$, $R_{loc,0}$ and m_0 are then the leading order values calculated in Sec. II. Specifically, the global correlation and response are to leading order $C_g=C_{loc,0}+C_{nl}$ and $R_g=R_{loc,0}+R_{nl}$ as before; we will not try to calculate $1/N$ corrections to these global quantities because these would require the subleading corrections to the non-local terms and thus an accurate treatment of quantities of $O(N^{-2})$.

In Eq. (9) we now need to keep the quadratic term in Δh_i. This gives for the magnetization

$$\frac{\partial m}{\partial t} = -m + \langle t_i\rangle = -m + \mathrm{th}(m) + \frac{\langle(\Delta h_i)^2\rangle}{2}\mathrm{th}''(m) + O(N^{-2}).$$

From the definition of h_i, $\langle(\Delta h_i)^2\rangle=\langle\Delta h_i\sigma_j\rangle$ for $j\neq i$, a quantity that we worked out in Eq. (A4). Expanding all quantities in the previous equation to order $O(N^{-1})$ then gives for the magnetization correction

$$\frac{\partial m_1}{\partial t} = -am_1 + \frac{1}{2}C_g(t,t)\mathrm{th}''(m_0).$$

This can be integrated but we will not give the explicit result here since it is not needed below.

For the correlations, by expanding Eq. (16) to $O(1/N)$ we arrive at

$$C_{loc,1}(t,t) = -2m_0(t)m_1(t). \tag{B1}$$

For $t\neq t_w$ we can use Eq. (A5) with $i=j$. Bearing in mind that $\langle h_i(t)\Delta\sigma_i(t_w)\rangle=C_{nl}(t,t_w)/N+O(N^{-2})$, the $O(N^{-1})$ terms give

$$\frac{\partial}{\partial t}C_{loc,1}(t,t_w) = -C_{loc,1}(t,t_w) + \mathrm{th}'(m_0)C_{nl}(t,t_w)$$

which integrates to

$$C_{\mathrm{loc},1}(t,t_{\mathrm{w}}) = e^{-(t-t_{\mathrm{w}})}C_{\mathrm{loc},1}(t_{\mathrm{w}},t_{\mathrm{w}}) + \int_{t_{\mathrm{w}}}^{t} dt' e^{-(t-t')}\mathrm{th}'(m_0(t'))C_{\mathrm{nl}}(t',t_{\mathrm{w}}). \quad \text{(B2)}$$

For the response, keeping the $(\Delta h_i)^2$ term in the expansion of (A12) gives

$$R_{ij}(t,t) = \delta_{ij}\left(\mathrm{th}'(m) + \frac{C_{\mathrm{g}}(t,t)}{2N}\mathrm{th}'''(m)\right) + O(N^{-2})$$

and the $O(N^{-1})$ terms show that the correction to the local instantaneous response is

$$R_{\mathrm{loc},1}(t,t) = m_1\mathrm{th}''(m_0) + \frac{1}{2}C_{\mathrm{g}}(t,t)\mathrm{th}'''(m_0). \quad \text{(B3)}$$

For the two-time response, Eq. (A8) with the $(\Delta h_i)^2$ term retained becomes

$$\frac{\partial m}{\partial t} + h\frac{\partial}{\partial t}R_{i1}(t,t_{\mathrm{w}}) = \mathrm{th}(m + h\delta\langle h_i\rangle) + \frac{1}{2}\langle(\Delta h_i)^2\rangle \times \mathrm{th}''(m + h\delta\langle h_i\rangle) - m - hR_{i1}(t,t_{\mathrm{w}}) + O(N^{-2}). \quad \text{(B4)}$$

To make progress we assume that the change of the variance $\langle(\Delta h_i)^2\rangle$ caused by the field h is $O(h/N^2)$ rather than $O(h/N)$ and can therefore be neglected. This can be made plausible by looking at the instantaneous response: the variance $\langle(\Delta\sigma_1)^2\rangle$ is changed by an amount of $O(h)$, while changes in all other covariances $\langle\Delta\sigma_i\Delta\sigma_j\rangle$ vanish. Thus $\langle(\Delta h_i)^2\rangle$ is indeed perturbed by a negligible amount $O(h/N^2)$, and one expects the response at later times to get no larger. We can therefore replace $\langle(\Delta h_i)^2\rangle$ by $C_{\mathrm{g}}(t,t)/N$ as before and regard it as h independent to the order in $1/N$ we are retaining. The $O(h)$ terms in Eq. (B4) then yield

$$\frac{\partial}{\partial t}R_{i1}(t,t_{\mathrm{w}}) = \left(\mathrm{th}'(m) + \frac{C_{\mathrm{g}}(t,t)}{2N}\,\mathrm{th}'''(m)\right)\delta\langle h_i\rangle - R_{i1}(t,t_{\mathrm{w}}) + O(N^{-2}).$$

For the local case of interest here, $i=1$, one has $\delta\langle h_i\rangle = R_{\mathrm{nl}}/N + O(N^{-2})$ from Eq. (A9). Therefore

$$\frac{\partial}{\partial t}R_{\mathrm{loc},0}(t,t_{\mathrm{w}}) + \frac{1}{N}\frac{\partial}{\partial t}R_{\mathrm{loc},1}(t,t_{\mathrm{w}}) = \mathrm{th}'(m_0)\frac{1}{N}R_{\mathrm{nl}}(t,t_{\mathrm{w}}) - R_{\mathrm{loc},0}(t,t_{\mathrm{w}}) - \frac{1}{N}R_{\mathrm{loc},1}(t,t_{\mathrm{w}}) + O(N^{-2}). \quad \text{(B5)}$$

The $O(N^{-1})$ terms give the desired equation of motion for the response correction

$$\frac{\partial}{\partial t}R_{\mathrm{loc},1}(t,t_{\mathrm{w}}) = -R_{\mathrm{loc},1}(t,t_{\mathrm{w}}) + \mathrm{th}'(m_0)R_{\mathrm{nl}}(t,t_{\mathrm{w}})$$

which integrates to

$$R_{\mathrm{loc},1}(t,t_{\mathrm{w}}) = e^{-(t-t_{\mathrm{w}})}R_{\mathrm{loc},1}(t_{\mathrm{w}},t_{\mathrm{w}}) + \int_{t_{\mathrm{w}}}^{t} dt' e^{-(t-t')}\mathrm{th}'(m_0(t'))R_{\mathrm{nl}}(t',t_{\mathrm{w}}). \quad \text{(B6)}$$

In evaluating the above general predictions we start with the unmagnetized case, where $m_0(t)=m_1(t)=0$ at all times. From Eq. (B1) the local equal-time correlation receives no correction, i.e., $C_{\mathrm{loc},1}(t,t)=0$, while for the response $R_{\mathrm{loc},1}(t,t)=-\beta^3C_{\mathrm{g}}(t,t)$ from Eq. (B3). For $T>T_{\mathrm{c}}$, we have seen that $C_{\mathrm{g}}(t,t)$ approaches its equilibrium value $1/a = 1/(1-\beta)$ exponentially. With $r(t)=\exp(-at)$ and Eq. (28) the global two-time correlation is then $C_{\mathrm{g}}(t,t_{\mathrm{w}})=a^{-1}\exp(-a\tau)$, while its local analog is given by Eq. (36) as $C_{\mathrm{loc},0}(t,t_{\mathrm{w}})=\exp(-\tau)$. Thus the correction $C_{\mathrm{loc},1}(t,t_{\mathrm{w}})$ to the local correlation is for long times

$$\beta\int_{t_{\mathrm{w}}}^{t} dt' e^{-(t-t')}(a^{-1}e^{-a(t'-t_{\mathrm{w}})} - e^{-(t'-t_{\mathrm{w}})}) = (1-\beta)^{-1}e^{-(1-\beta)\tau} - e^{-\tau}[(1-\beta)^{-1} + \beta\tau]. \quad \text{(B7)}$$

Combining this with the leading order term gives the result (48) discussed in the main text. To work out the corresponding correction to the response one notes from Eq. (B3) that the equal-time value $R_{\mathrm{loc},1}(t,t)=-\beta^3C_{\mathrm{g}}(t,t)$ converges exponentially to $-\beta^3/a$. Also, the global response is $R_{\mathrm{g}}(t,t_{\mathrm{w}}) = \beta\exp(-a\tau)$ from Eq. (29) and the local one $R_{\mathrm{g}}(t,t_{\mathrm{w}}) = \beta\exp(-\tau)$ according to Eq. (37). By inserting these results into Eq. (B6) one finds

$$R_{\mathrm{loc},1}(t,t_{\mathrm{w}}) = \beta e^{-(1-\beta)\tau} - e^{-\tau}\left(\frac{\beta^3}{1-\beta} + \beta + \beta^2\tau\right)$$

and it is easy to check that this is related by FDT to the correlation correction (B7) as it should be.

Next consider the out-of-equilibrium dynamics at criticality ($T=1$) starting from zero magnetization. For the correlation correction we use that $C_{\mathrm{g}}(t,t_{\mathrm{w}})=C_{\mathrm{g}}(t_{\mathrm{w}},t_{\mathrm{w}})$ [a consequence of Eq. (28) together with $r(t)=1$], while $C_{\mathrm{loc},0}(t,t_{\mathrm{w}}) = \exp(-\tau)$ as before. Then Eq. (B2) results in

$$C_{\mathrm{loc},1}(t,t_{\mathrm{w}}) = \int_{t_{\mathrm{w}}}^{t} dt' e^{-(t-t')}[C_{\mathrm{g}}(t_{\mathrm{w}},t_{\mathrm{w}}) - e^{-(t'-t_{\mathrm{w}})}] = C_{\mathrm{g}}(t_{\mathrm{w}},t_{\mathrm{w}})(1-e^{-\tau}) - \tau e^{-\tau}$$

and adding the leading order term gives (49). The calculation for the response correction proceeds similarly, with $R_{\mathrm{g}}(t,t_{\mathrm{w}})=1$ from Eq. (29) and $R_{\mathrm{loc}}(t,t_{\mathrm{w}})=\exp(-\tau)$, and leads to

$$R_{\mathrm{loc},1}(t,t_{\mathrm{w}}) = -e^{-\tau}C_{\mathrm{g}}(t_{\mathrm{w}},t_{\mathrm{w}}) + 1 - e^{-\tau} - \tau e^{-\tau}$$

and hence Eq. (50).

The final case of interest is the dynamics at criticality but with $m(0)\neq 0$. To obtain the leading contribution at long times to the correction $C_{\mathrm{loc},1}(t,t_{\mathrm{w}})$, consider the integral term in Eq. (B2). The nonlocal correlations are $C_{\mathrm{nl}}(t,t_{\mathrm{w}})=(t_{\mathrm{w}}/2) \times(t_{\mathrm{w}}/t)^{3/2}-\exp(-\tau)$. The second term makes a contribution which is at most $O(1)$, and exponentially suppressed for $\tau \gg 1$. The first term, on the other hand, contributes

$$\int_{t_w}^{t} dt' e^{-(t-t')} C_g(t',t_w) \approx C_g(t,t_w) \tag{B8}$$

where we have used that for $t_w \gg 1$ the factor $C_g(t',t_w)$ in the integrand varies negligibly in the region $t-t'=O(1)$ that contributes significantly, and the exponential integrates to unity for $\tau \gg 1$. Taking into account that the first term in Eq. (B2) is also exponentially suppressed, Eq. (B8) gives the leading contribution to $C_{loc,1}(t,t_w)$ for $\tau \gg 1$ and $t_w \gg 1$. For the response, very similar arguments show that the leading contribution to the integral term in Eq. (B6) is simply $R_g(t,t_w)$; for time differences $\tau \gg 1$ the remainder of the integral and the first term in Eq. (B6) can be disregarded. This leads to the results (50) and (51) given in the main text.

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